Journal of Engineering Mathematics **26**: 525–537, 1992.

On the dual solutions associated with boundary-layer equations in a corner

A. RIDHA
Université P. et M. Curie (Paris 6), Laboratoire de Modélisation en Mécanique, CNRS-URA 229, Tour 66, 4 Place Jussieu, 75252 Paris Cedex 05, France
Laboratoire de Mécanique, Université de Caen, 14032 Caen Cedex, France

Received 20 September 1991; accepted in revised form 24 April 1992

Abstract. The dual solutions of two coupled third degree non-linear ordinary differential equations associated with the incompressible viscous laminar flow along a corner are considered. It is shown (through the numerical solution) that dual solutions occur in the interval $\beta_b \leqslant \beta \leqslant 1.1211$ for the Falkner–Skan parameter β with the bifurcation taking place at the regular turning point β_b. In the neighbourhood of the latter it is also shown that in such a case it is appropriate to expand the solution in powers of $(\beta - \beta_b)^{1/2}$ with the dual solutions branching out from the single solution at β_b. Then, on considering a simple transient problem (which provides an exact solution of the Navier–Stokes equations when $\beta = 1.0$) it is found that the branch having the greatest value of the wall shear stress (for a given β) is stable while the other is unstable, the bifurcation point being the point of exchange of stability.

1. Introduction

The laminar viscous incompressible flow along a streamwise corner formed by the intersection of two semi-infinite flat plates with coplanar leading edges has attracted considerable attentions for many decades. The viscous flow in the vicinity of the intersection line is inherently three-dimensional and is often referred to as the corner boundary layer (CBL). Such a corner geometry (see Fig. 1a) has provided a simple corner model tenable to theoretical analysis. To the author's knowledge theoretical work on this problem was inaugurated in the mid-thirties by Loitsianskii et al. [1, 2] and followed ever since by many investigations some of which are found in [3]–[10].

For the purpose of the present article the flow may be envisaged divided into four regions as shown in Fig. 1b (see also [4] and [8]). Region I is defined by $x_1 > 0$, $x_2 = O(\mathrm{Re}^{-1/2})$ and

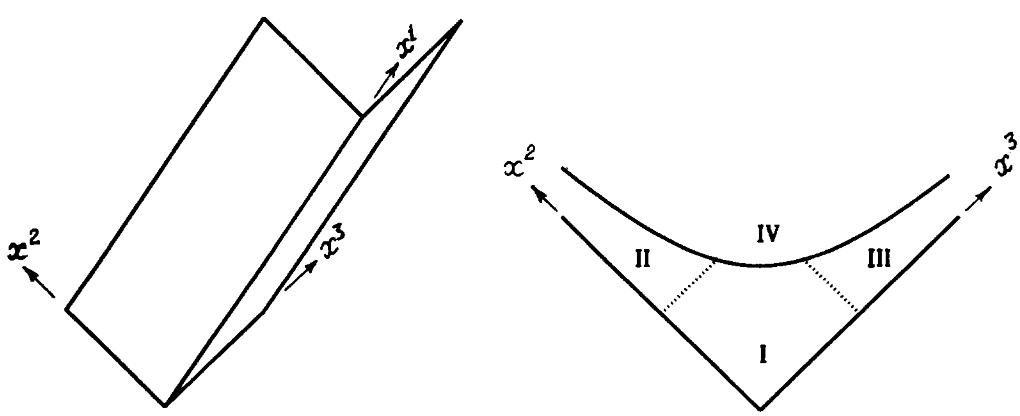

Fig. 1. Corner flow configuration.

$x_3 = O(\mathrm{Re}^{-1/2})$ whereas in II (respectively III) $x_1 > 0$, $x_2 = O(1)$ and $x_3 = O(\mathrm{Re})^{-1/2}$ (respectively $x_1 > 0$, $x_3 = O(1)$ and $x_2 = O(\mathrm{Re})^{-1/2}$). Here, $\mathrm{Re} = Ul/\nu$ ($\gg 1$) is the Reynolds number, U and l designate reference velocity and length scales and ν the kinematic viscosity. With regards to the inviscid flow it may, to first approximation, be represented by IV where $x_1 > 0$, $x_2 = O(1)$ and $x_3 = O(1)$ which reflects that region I is to 'blend' with IV as $(x_2, x_3)\,\mathrm{Re}^{1/2} \to \infty$, $x_1 > 0$. This division, though may appear similar to that proposed in [4] or [8], is in fact different. In our analysis the two-dimensional streamwise component is obtained at the outer limit of II and III whereas it is obtained at the inner limit of these same regions in [4] and [8]. In the present problem the latter situation arises only for $\beta = 0$ and only for one of the dual solutions at this value. This difference arises from the fact that the matching requirements, in our case, between I, II and III with IV amounts to satisfying the symmetry conditions only. Thus, to complete the boundary layer picture we need to add a further inviscid region to effect the matching of the boundary layer in II and III with the strictly two-dimensional flow prevailing at the outer limits of these regions. In this context it is worth mentioning the work of Smith [9] on the three-dimensional stagnation point flow into a corner. Thereat, it was necessary to divide the corner layer into five regions (four regions in Smith's notation) which is in essence the same division used to analyse the present corner problem.

Most investigations (see for example [1]–[8]) were confined to zero streamwise pressure gradient situations where theoretical and experimental results obtained so far exhibit non-negligible quantitative differences depending on both the theoretical and experimental models used in each. Zamir [11] discussed this issue in some details. What is of direct interest to us here is the central conclusion which emerges from Zamir's work. This may be stated as follows. For a rectangular corner configuration the viscous flow at Reynolds numbers greater than 10^4 gives rise to CBL which is stable only with some favourable pressure gradient. Zamir notes also that the CBL becomes progressively unstable as the gradient is reduced to zero.

These findings prompted the present author [10] to consider the problem with non-zero pressure gradient parallel to the corner line arising from a mainstream having a velocity component (in the x_1 direction) proportional to $x_1^{\beta/(2-\beta)} + O(\mathrm{Re}^{-1/2})$ where β is the Falkner–Skan parameter. The analysis of the boundary conditions (see §2, Eqs (1)) controlling the CBL (region I) prevailing in II and III revealed that the said conditions are non-unique; i.e. dual solutions of the relevant equations are obtained for the range $-0.03678 \leqslant \beta \leqslant 1.1211$. (See Smith [20] for other examples of non-uniqueness in boundary layers.) This is a new result worthy of more examination in the light of the said differences between theory and experiments. The purpose of the present paper is, therefore, to indulge into further investigation of Eqs (1) and examine which of the two solutions is likely to be obtained in practice.

2. Posing the problem

The (said) boundary conditions, prevailing in region II (or III) and governing the boundary-layer at I, are composed of sets of equations resulting from an asymptotic series expansion the leading order of which gives the following system of equations

$$\left.\begin{aligned} f''' + [(2-\beta)h + f]f'' + \beta(1-f'^2) = 0\,, \\ h''' + [(2-\beta)h + f]h'' + [2(1-\beta)f' - (2-\beta)h']h' = \frac{\beta(4-3\beta)}{4(2-\beta)} \end{aligned}\right\} \tag{1a}$$

together with the boundary conditions

$$\left.\begin{aligned} &f(0) = h(0) = f'(0) = h'(0) = 0\,, \\ &\eta \to \infty\,; \quad f' \to 1\,, \quad h' \to -\frac{\beta}{2(2-\beta)}\,. \end{aligned}\right\} \tag{1b}$$

Here, the prime denotes differentiation with respect to the independent space variable η. (For more details on the derivation of (1) see [10].) The variables (f, h) are suitable 'stream functions' with (f', h') denoting respectively the main flow velocity parallel to the corner line and a convenient (but partial) representation of one of the secondary flow velocity components in the x_2 (or x_3) direction which corresponds to region II (or III). Although system (1) was derived on consideration of a rectangular corner situation, it is in fact valid for a corner of an arbitrary angle with suitable modifications. Further, when the corner angle is close enough to π it can be shown that Eqs (1) are valid at and near the symmetry plane and provide an exact solution of the Navier–Stokes equations for $\beta = 1$. The proof of this requires some space and has no immediate bearing on the purpose of the present paper; we shall report its details in a future publication.

In [10] the dual solutions of (1) were obtained in the interval $-0.03678 \leqslant \beta \leqslant 0.285$ but in fact with more severe and comprehensive numerical investigations we have found that the said interval extends to $\beta \leqslant 1.1211$. For convenience we shall refer to the solution with the higher value of the wall shear stress $f''(0)$, for a given value of β, as the upper-branch solution (say, $\mathbb{F}_u$) and in consequence that with the smaller value the lower-branch solution ($\mathbb{F}_l$). Here, $\mathbb{F}$ denotes $\mathbb{F} = [f, h]^T$.

Unlike the Falkner–Skan equation, system (1) admits no reverse flow in the streamwise direction (i.e., $f''(0) > 0$) while the secondary flow (which is related to h') undergoes a reversal ($h''(0) < 0$) throughout the lower branch. In the special case of $\beta = 0$ the Blasius solution (f) is obtained on the lower-branch together with $h = 0$ at $0 \leqslant \eta < \infty$. The second solution is found to pertain to the upper-branch.

In this paper, we shall, at first, present some numerical results of system (1) in the range $-0.03678 \leqslant \beta < 2$. Thereafter we determine the bifurcation point β_b and examine the solution in its neighbourhood. Whence we find that the perturbation to the solution is $O((\beta - \beta_b)^{1/2})$. Here, the dual solutions branch out from the single solution at β_b with the upper-solution stemming out from the positive sign (arising from taking a square root) whereas the lower one from the negative sign. Our method in this regard is similar to that used by Merkin [12].

Having two solutions at hand makes it somewhat imperative to seek an answer as to which of the two may be obtained in practice. With the knowledge that both solutions display physically acceptable features, with the Blasius solution being a special case, it becomes rather intriguing to investigate this matter in some detail. As often with equations having more than one solution under given conditions [12, 19] the approach used here to tackle this question is to consider the transient problem

$$\frac{\partial^2 \mathcal{F}}{\partial \eta\, \partial \tau} = \mathcal{D}\mathcal{F} + \mathcal{C} \tag{2a}$$

where

$$\left.\begin{aligned}
&\mathscr{F}=\begin{bmatrix} F(\eta,\tau) \\ H(\eta,\tau) \end{bmatrix}, \qquad \mathscr{C}=\begin{bmatrix} \beta \\ \dfrac{\beta(4-3\beta)}{4(2-\beta)} \end{bmatrix}, \\
&\mathscr{D}=\begin{bmatrix} \mathscr{A} & O \\ O & \mathscr{A}+(2-\beta)(F'-H')\dfrac{\partial}{\partial\eta} \end{bmatrix}, \\
&\mathscr{A}=\frac{\partial^3}{\partial\eta^3}+((2-\beta)H+F)\frac{\partial^2}{\partial\eta^2}-\beta F'\frac{\partial}{\partial\eta}
\end{aligned}\right\} \tag{2b}$$

and (F, H) standing for transient functions corresponding to (f, h). A prime in (2b) denotes partial differentiation with respect to η. The boundary conditions for $\tau > 0$ are the same as in (1b) and for $\tau = 0$ it is sufficient to use the steady state solution $\mathbb{F}_l$ to investigate the above evolution problem. It is noteworthy to underline at this point that Eqs (2) offer a new exact solution of the Navier–Stokes equations when $\beta = 1$. In this respect the investigation of (2) assumes considerable interest in this vicinity. Throughout the dual solution range of β we have found that the solution $\mathscr{F}(\eta, \tau)$ approaches the upper solution $\mathbb{F}_u(\eta)$ as $\tau \to \infty$. The reason behind this behaviour becomes apparent when we subject the solution of (2) to small disturbances and examine the results for large τ. The disturbance is $(\mathscr{F}(\eta, \tau) - \mathbb{F}(\eta))$ and is of $O(e^{-\lambda\tau})$ where λ is governed by a linear eigenvalue problem involving $\mathbb{F}_l$ or $\mathbb{F}_u$. Solving for the *smallest* eigenvalue λ_l in the β-λ space we find that λ_l is always negative for $\mathbb{F}_l$ and positive for $\mathbb{F}_u$ which shows that for our problem $\mathbb{F}_u$ is the stable solution whereas $\mathbb{F}_l$ is the unstable one. The behaviour is, of course, in complete conformity with the numerical solution of the evolution problem (2). These results have there correspondence in situations where bifurcation of the kind occuring here takes place as, for example, in the case of reference [12].

3. Some numerical results

System (1) was solved numerically with double precision by using the shooting techniques proposed by Cebeci and Keller [15]. We have considered a solution as satisfactory once $f'(\infty)$ and $h'(\infty)$ attain their respective values to a precision of 10^{-8}. In Fig. 2 are shown the primary and secondary wall shear stresses $f''(0)$ and $h''(0)$ together with the wall shear stress $F''_{fs}(0)$ corresponding to the Falkner–Skan equation added for reference. A point worth underlining in these results is that the Blasius solution lies on the lower-branch. It is also of interest to note here that in previous works [3–8] the solution obtained for f at zero pressure gradient pertained to the Blasius solution. We recall that in experimental works [16] 'venturing' to obtain a stable laminar CBL zero streamwise pressure gradient, the wall shear stress in regions corresponding to II (or III) overshoots markedly the Blasius value (before tending asymptotically to it further away from the corner line). Such a behaviour lends support to our results in this region since on the upper-branch we find the solution with the higher value of the wall shear stress and therefore we may look upon the experimental results as a 'verification' of the existence of the present upper-branch solution.

In conformity with the above and unlike $F''_{fs}(0)$ which possesses negative values (reflecting

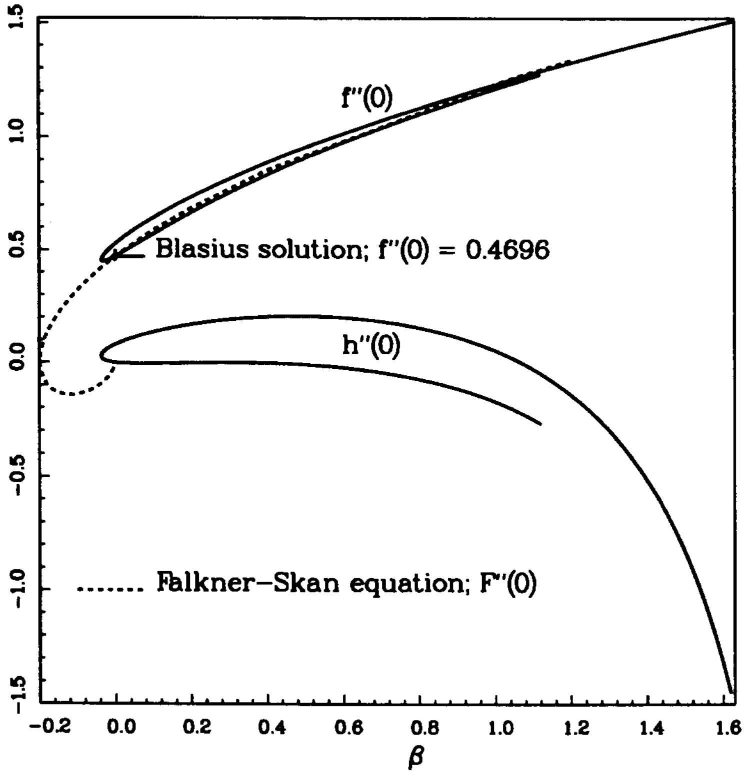

Fig. 2. Wall shear stress variation with β.

the existence of reverse flow) for $-0.19884 \leqslant \beta < 0$ on its lower-branch, $f''(0)$ *is* always positive. On the other hand we remark that $h''(0)$ becomes negative for all values of β on the lower-branch as well as for $\beta \geqslant 1.0716$ on the upper branch. This is rather a complicated behaviour which prompts us to seek the solution of the full corner boundary layer equations derived in [10]. This solution is currently under investigation and will be reported in due time.

Some results for f' and h' are depicted in Fig. 3 where profiles of these variables are shown for different values of β. We observe that the solution at β_b lies between the lower- and upper-branch solutions for $\beta = 0$. Further, it is worth noting at this point that, for $\beta = 1$, we have, in essence, the same equations obtained in reference [9] – though differently formulated – on treating the problem of the three-dimensional stagnation point flow into a corner. In the absence of the parameter β there was no mechanism in [9] which could bring into light the dual solutions as in the present work; only one solution was obtained and was the upper-branch one.

4. The solution near the bifurcation point β_b

From the numerical results it is evident that the bifurcation point in our problem is a regular turning point. The neighbourhood of such points may be approximated by a polynomial of the second degree. By this we mean that it is appropriate to expand the solution in powers of the square-root of some suitable small parameter. Indeed, such was the situation in the case

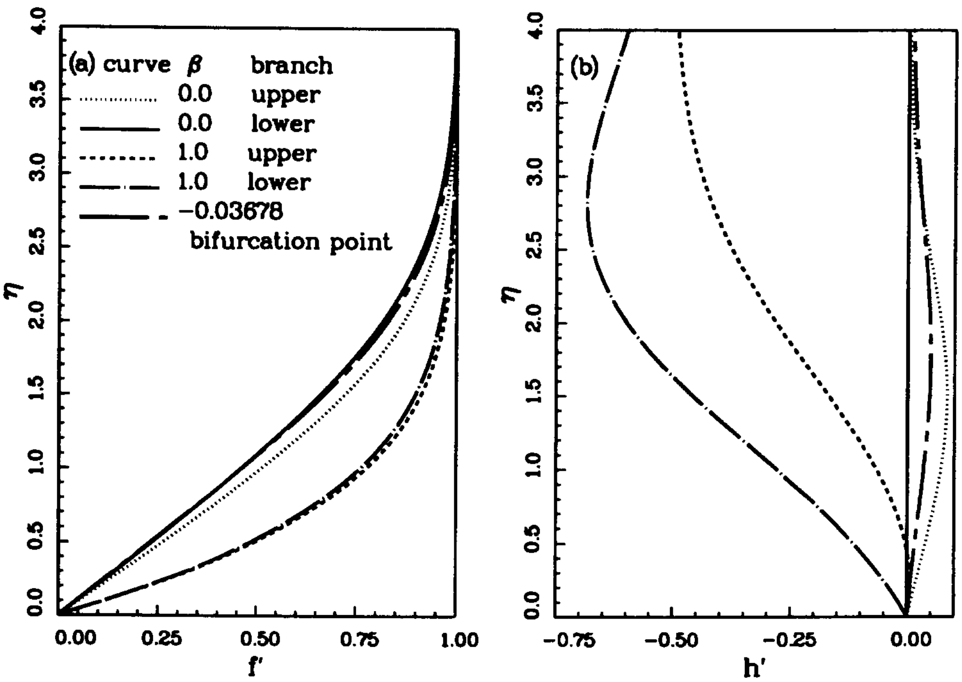

Fig. 3. (a) Profiles of the streamwise velocity f'; (b) profiles of h'.

of the Falkner–Skan equation [17]. We shall therefore seek to find the bifurcation point β_b of the dual solutions by setting $\varepsilon = \beta - \beta_b$ and expanding f and h in powers of $\varepsilon^{1/2}$ (for $0 < \varepsilon \ll 1$). In so doing our approach is analogous to that of Mahmood and Merkin [13]. Thus the required expansion is

$$\left.\begin{aligned} f &= f_0 + \varepsilon^{1/2} f_1 + \varepsilon f_2 + \cdots , \\ h &= h_0 + \varepsilon^{1/2} h_1 + \varepsilon h_2 + \cdots \end{aligned}\right\} \tag{3}$$

with (f_0, h_0) being the solution of system (1) at β_b. The system of equations corresponding to $O(\varepsilon^{1/2})$ is expressed by the following homogeneous problem

$$\left.\begin{aligned} f_1''' + [(2-\beta_b)h_0 + f_0]f_1'' + [(2-\beta_b)h_1 + f_1]f_0'' - 2\beta_b f_0' f_1' = 0 , \\ h_1''' + [(2-\beta_b)h_0 + f_0]h_1'' + [(2-\beta_b)h_1 + f_1]h_0'' + \\ [2(1-\beta_b)f_0' - (2-\beta_b)h_0']h_1' + [2(1-\beta_b)f_1' - (2-\beta_b)h_1']h_0' = 0 \end{aligned}\right\} \tag{4a}$$

subject to the boundary conditions

$$f_1(0) = h_1(0) = f_1'(0) = h_1'(0) = f_1'(\infty) = h_1'(\infty) = 0 . \tag{4b}$$

To solve system (4) we apply the extra boundary condition $f_1''(0) = 1$ to force a non-trivial solution and consider the value of β_b as an extra parameter which must be determined. This is rather like the inverse problem approach used in [13], [14] and [15]. In fact, it is the solution of this homogeneous problem which furnishes the value of β_b. As regards the general solution of this problem we have $f_1 = \kappa \bar{f}$, $h_1 = \kappa \bar{h}$ where $\bar{f}$, $\bar{h}$ denotes that solution for which $\bar{f}''(0) = 1$. Now let us consider the equations at order $O(\varepsilon)$ which are

$$\left.\begin{aligned}
&f_2''' + [(2-\beta_b)h_0 + f_0]f_2'' + [(2-\beta_b)h_2 + f_2]f_0'' - 2\beta_b f_0' f_2' = \\
&\quad h_0 f_0'' - 1 + f_0'^2 - \kappa^2[(2-\beta_b)\bar{h} + \bar{f}]\bar{f}'' + \beta_b \kappa^2 \bar{f}'^2 , \\
&h_2''' + [(2-\beta_b)h_0 + f_0]h_2'' + [(2-\beta_b)h_2 + f_2]h_0'' + \\
&[2(1-\beta_b)f_0' - (2-\beta_b)h_0']h_2' + [2(1-\beta_b)f_2' - (2-\beta_b)h_2']h_0' = \\
&\quad h_0 h_0'' + 2f_0' h_0' - h_0'^2 - \kappa^2[(2-\beta_b)\bar{h} + \bar{f}]\bar{h}'' - \\
&\quad \kappa^2[2(1-\beta_b)\bar{f}' - (2-\beta_b)\bar{h}']\bar{h}' - \frac{3\beta_b^2 - 12\beta_b + 8}{4(2-\beta_b)^2}
\end{aligned}\right\} \tag{5a}$$

with the boundary conditions

$$f_2(0) = h_2(0) = f_2'(0) = h_2'(0) = f_2'(\infty) = 0 , \qquad h_2'(\infty) = -1/(2-\beta_b)^2 . \tag{5b}$$

We note that as $\eta \to \infty$, $f_0 \sim \eta + \delta_f +$ exponentially small terms (E.S.T) and $h_0 \sim (-\beta_b/2 \times (2-\beta_b))\eta + \delta_h +$ E.S.T. where δ_f and δ_h are constants. If we set

$$\xi = \eta + (\delta_f + (2-\beta_b)\delta_h)/(1-\beta_b/2) ,$$

Eqs (5) would then, in general, have the solution

$$\left.\begin{aligned}
f_2' &\sim A\xi^{4\beta_b/(2-\beta_b)}\left(1 + \frac{4\beta_b(5\beta_b - 2)}{(2-\beta_b)^3}\,\xi^{-2} + \cdots\right) \\
h_2' &\sim -\frac{1}{(2-\beta_b)^2} + \frac{\beta_b(1-\beta_b)}{(2+\beta_b)(2-\beta_b)}\,f_2' + B\xi^{-2}\left(1 + \frac{6}{(2-\beta_b)}\,\xi^{-2} + \cdots\right)
\end{aligned}\right\} \tag{6}$$

for ξ large where a prime denotes, here, differentiation with respect to ξ, A and B being integration constants.

In order to solve Eqs (5) we proceed, similar to [13], to construct the following two integrals:

(a) (f_a, h_a) with $f_a''(0) = h_a''(0) = 0$ which satisfies Eqs (5) with $\kappa = 1$ and the remaining terms on the right-hand side being omitted;
(b) (f_b, h_b) with $f_b''(0) = h_b''(0) = 0$ which satisfies Eqs (5) with $\kappa = 0$. Further, we also construct two complementary functions as follows:
(c) (f_c, h_c) with $f_c''(0) = 1$, $h_c''(0) = 0$;
(d) (f_d, h_d) with $f_d''(0) = 0$, $h_d''(0) = 1$.

The full solution may therefore be written in the form

$$\begin{bmatrix} f_2' \\ h_2' \end{bmatrix} = \begin{bmatrix} f_a' & f_b' \\ h_a' & h_b' \end{bmatrix}\begin{bmatrix} \kappa^2 \\ 1 \end{bmatrix} + \begin{bmatrix} f_c' & f_d' \\ h_c' & h_d' \end{bmatrix}\begin{bmatrix} \gamma \\ \mu \end{bmatrix} - \begin{bmatrix} 0 \\ 1/(2-\beta_b)^2 \end{bmatrix} \tag{7}$$

for some constants γ and μ. Now, from considerations of Eqs (6) this solution behaves in general like

$$\begin{bmatrix} f_2' \\ h_2' \end{bmatrix} = \begin{bmatrix} A_a & A_b \\ \xi^{-2}B_a & \xi^{-2}B_b \end{bmatrix}\begin{bmatrix} \kappa^2 \\ 1 \end{bmatrix} + \begin{bmatrix} A_c & A_d \\ \xi^{-2}B_c & \xi^{-2}B_d \end{bmatrix}\begin{bmatrix} \gamma \\ \mu \end{bmatrix} - \begin{bmatrix} 0 \\ 1/(2-\beta_b)^2 \end{bmatrix} \tag{8}$$

as $\eta \to \infty$. Therefore, in order to satisfy the boundary conditions we should have

$$\begin{bmatrix} A_a & A_b \\ B_a & B_b \end{bmatrix}\begin{bmatrix} \kappa^2 \\ 1 \end{bmatrix} + \begin{bmatrix} A_c & A_d \\ B_c & B_d \end{bmatrix}\begin{bmatrix} \gamma \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9a}$$

for a suitable choice of γ and μ. Now recalling that to order $O(\varepsilon^{1/2})$ we have a non-trivial solution (which infers that $\kappa \neq 0$) then we must have non-zero values for μ and γ such that

$$\begin{vmatrix} A_a & A_b \\ B_a & B_b \end{vmatrix} = 0 \,, \qquad \begin{vmatrix} A_c & A_d \\ B_c & B_d \end{vmatrix} = 0 \,. \tag{9b}$$

It is evident then that Eqs (9) are compatible if and only if

$$\kappa^2 = \frac{B_b A_c - A_b B_c}{B_c A_a - A_c B_a} \,. \tag{10}$$

It goes without saying that Eq. (10) has two solutions which are $\mp \kappa_b$ (say), and as a consequence we have near the bifurcation point β_b the following expansion

$$\left.\begin{aligned} f''(0) &= f_0''(0) \mp \kappa_b(\beta_b - \beta)^{1/2} + \cdots \,, \\ h''(0) &= h_0''(0) \mp \kappa_b \bar{h}''(0)(\beta_b - \beta)^{1/2} + \cdots \,, \end{aligned}\right\} \tag{11}$$

where the "+" sign pertains to the start of the upper-branch behaviour while the "−" sign to the lower-branch. In the present case we have found that $|\kappa_b| = 0.2269$ and $\bar{h}''(0) = 0.0282$. Figure 4 depicts the wall shear stress behaviour given in (11) together with the corresponding numerical solution of system (1) near the bifurcation point κ_b. In this neighbourhood a good agreement is observed with the numerical solution.

5. The transient problem

Here we shall consider the evolution with time of the solution of Eq. (1), as expressed in (2). We use finite-differences to approximate the time derivative and the Crank–Nicholson method to express the operator $\mathscr{D}$ as shown below

$$\frac{(\partial \mathscr{F}/\partial \eta)^{n+1} - (\partial \mathscr{F}/\partial \eta)^n}{\Delta \tau} = \frac{1}{2}(\mathscr{D}^{n+1} + \mathscr{D}^n)\mathscr{F}^{n+1} + \mathscr{C} \tag{12}$$

where the superscript n refers to the time step $n\,\Delta\tau$. The boundary conditions are as follows

$$\left.\begin{aligned} &\tau = 0 \,; && \mathscr{F}(\eta, 0) = \mathbb{F}_l \,, \\ &\tau > 0 \,; && F(0,\tau) = H(0,\tau) = F'(0,\tau) = H'(0,\tau) = 0 \,, \\ &\eta \to \infty \,; && F'(\infty,\tau) = 1 \,, \quad H'(\infty,\tau) = -\frac{\beta}{2(2-\beta)} \end{aligned}\right\} \tag{13}$$

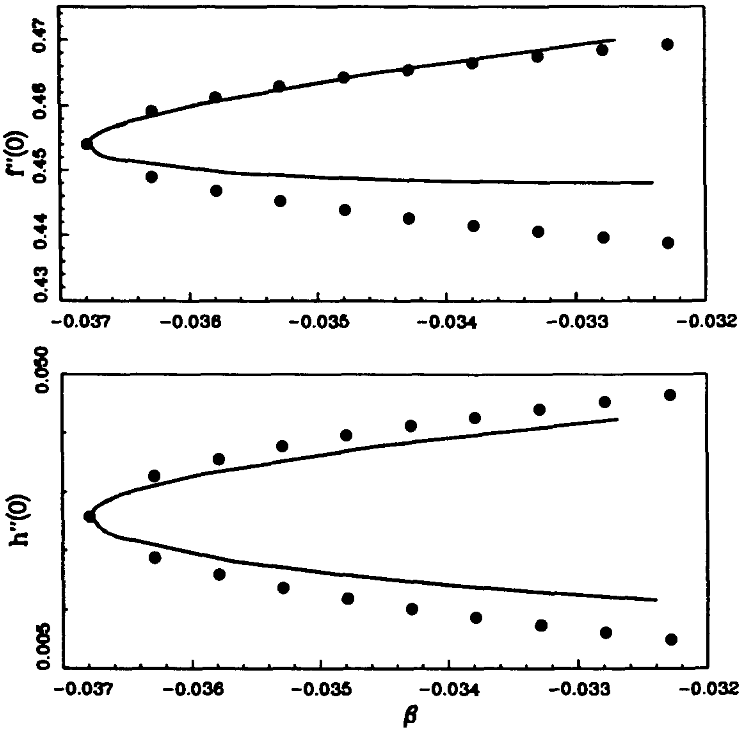

Fig. 4. The wall shear stress evolution near the bifurcation point β_b; the solid line pertains to the numerical solution of Eq. (1) and the solid circles to Eq. (11).

where here again a prime denotes differentiation with respect to η. We note that it is sufficient for our purpose to use the lower-branch solution as our initial condition. The scheme of the solution, at each time step, is once again the shooting method of reference [14] applied with double precision and 10^{-8} accuracy. Throughout the dual solutions space of β and as $\tau \rightarrow \infty$ the upper branch solution is obtained. Recalling that for $\beta = 1$ Eqs (1) represent an exact solution of the Navier–Stokes equations, we give in Fig. 5 the transient behaviour for this case as well as that for $\beta = 0$ as depicted in terms of $F''(0, \tau)$ and $H''(0, \tau)$. This behaviour reflects that the lower-branch solution is unstable and the error accumulation in the course of computation becomes sufficiently large enough, at some stage, to force the solution to leave the lower-branch basin onto the upper-branch. It is worth noting here that the same behaviour was obtained when system (1), after some reformation, we used in [18] as a test problem to propose a new numerical algorithm for non-linear boundary-value problems.

Such a "sliding" behaviour from one branch to the other is also observed in [12] where dual solutions were obtained for a mixed convection problem in a porous medium. To investigate the reasons behind this tendency of the transient problem we impose a small disturbance $(F_1(\eta, \tau), H_1(\eta, \tau))$ on the solution by writing $F(\eta, \tau) = F_0(\eta) + F_1(\eta, \tau)$ and $H(\eta, \tau) = H_0(\eta) + H_1(\eta, \tau)$ with F_0 and H_0 corresponding to the solution of system (1). Whence we have the linearized problem

$$\frac{\partial^2 \mathcal{F}_1}{\partial \eta \, \partial \tau} = \mathcal{D}_0 \mathcal{F}_1 + \mathcal{D}_1 \mathcal{F}_0 \tag{14}$$

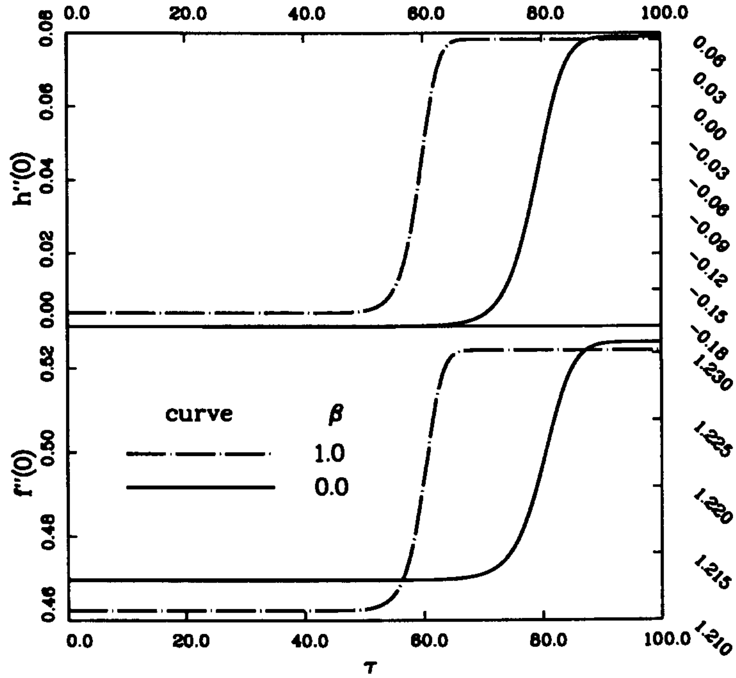

Fig. 5. The transient behaviour of the Wall shear stress for $\beta = 1$ and $\beta = 0$.

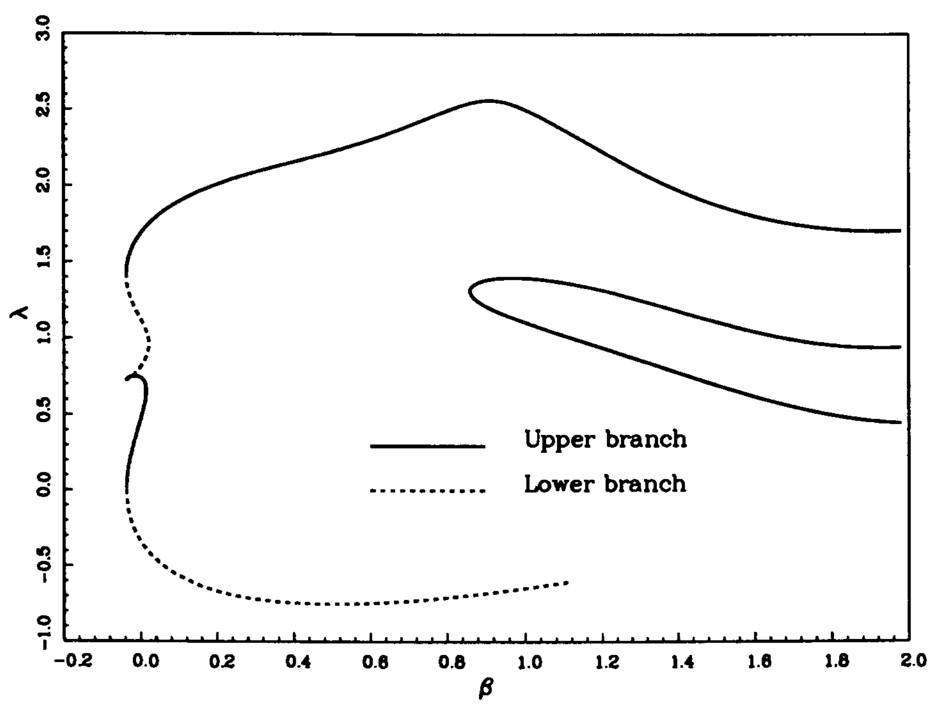

Fig. 6. The evolution of the eigenvalue λ with β.

where $\mathscr{D}_0$ and $\mathscr{D}_1$ are equivalent to the operator $\mathscr{D}$ with (F, H) replaced by (F_0, H_0) and (F_1, H_1) respectively. Also, we have $\mathscr{F}_0 = [F_0, H_0]^T$ and $\mathscr{F}_1 = [F_1, H_1]^T$.

In general, some of the solutions which bifurcate are stable and some are unstable [19]. The stability of such solutions may be examined by reverting to the linearized problem the solution of which is sought by setting $(F_1, H_1) = e^{-\lambda\tau}(\varphi(\eta), \psi(\eta))$ for some constant λ. Thus, we find that the disturbance (F_1, H_1) grows or decays according to whether $\lambda < 0$ or $\lambda > 0$ respectively. Written in full, the linearized problem reads

$$\left.\begin{aligned}
&\varphi''' + ((2-\beta)H_0 + F_0)\varphi'' + ((2-\beta)\varphi + \psi)F_0'' - 2\beta F_0'\varphi + \lambda\varphi' = 0\,,\\
&\psi''' + ((2-\beta)H_0 + F_0)\psi'' + ((2-\beta)\varphi + \psi)H_0'' +\\
&(2(1-\beta)F_0' - (2-\beta)H_0')\psi' + (2(1-\beta)\varphi' - (2-\beta)\psi')H_0' + \lambda\psi' = 0
\end{aligned}\right\} \tag{15}$$

subject to the boundary conditions

$$\varphi(0) = \psi(0) = \varphi'(0) = \psi'(0) = \varphi'(\infty) = \psi'(\infty) = 0\,. \tag{16}$$

The homogeneous Eqs (15) represent a linear eigenvalue problem in λ. It remains now to determine the smallest eigenvalue λ for each solution. To this end we proceed as in [12]. On considering Eqs (15) for large ξ (ξ having the same definition as in §4) we obtain the following asymptotic approximation

$$\left.\begin{aligned}
&\varphi''' + \frac{1}{2}(2-\beta)\xi\varphi'' + (\lambda - 2\beta)\varphi' = 0\,,\\
&\psi''' + \frac{1}{2}(2-\beta)\xi\psi'' + \left(\lambda + \frac{1}{2}(4-3\beta)\right)\psi' - \frac{\beta(1-\beta)}{(2-\beta)}\varphi' = 0\,.
\end{aligned}\right\} \tag{17}$$

The solution of (17) is

$$\left.\begin{aligned}
&\varphi' \sim C\xi^{-\frac{2\lambda-4\beta}{2-\beta}} + D\xi^{\frac{2\lambda-3\beta-2}{2-\beta}} e^{-\frac{2-\beta}{4}\xi^2}\,,\\
&\psi' \sim \frac{2\beta(1-\beta)}{(2-\beta)(4+\beta)}\varphi' + G\xi^{-\frac{4-3\beta+2\lambda}{2-\beta}} + J\xi^{\frac{2(1-\beta+\lambda)}{2-\beta}} e^{-\frac{2-\beta}{4}\xi^2}
\end{aligned}\right\} \tag{18}$$

where C, D, G and J are integration constants. Further, to force non-trivial solutions for Eqs (15) (subject to (16)) we carry out their integration with non-zero boundary conditions either for $\varphi''(0)$ or for $\psi''(0)$. Thus we may integrate (15) twice with the following additional boundary conditions,

1. $\varphi''(0) = 1$, $\psi''(0) = 0$,
2. $\varphi''(0) = 0$, $\psi''(0) = 1$.

Accordingly we have for the full solution

$$\begin{bmatrix}\varphi'\\ \psi'\end{bmatrix} = \begin{bmatrix}\varphi_1' & \varphi_2'\\ \psi_1' & \psi_2'\end{bmatrix}\begin{bmatrix}\gamma\\ \mu\end{bmatrix}, \tag{19a}$$

for some arbitrary constants γ and μ. Return now to (18) from which we observe that for large ξ the general solution assumes the following form

$$\begin{bmatrix} \varphi' \\ \psi' \end{bmatrix} \sim \begin{bmatrix} C_1(\lambda) & C_2(\lambda) \\ G_1(\lambda)g(\xi) & G_2(\lambda)g(\xi) \end{bmatrix} \begin{bmatrix} \gamma \\ \mu \end{bmatrix} + \frac{2\beta(1-\beta)}{(2-\beta)(4+\beta)} \begin{bmatrix} 0 \\ \varphi' \end{bmatrix} \tag{19b}$$

where $g(\xi) = \xi^{-\frac{4-3\beta+2\lambda}{2-\beta}}$. Since solutions (φ_1, ψ_1) and (φ_2, ψ_2) are non-trivial then we must have non-zero values for γ and μ. This requires that

$$\begin{vmatrix} C_1(\lambda) & C_2(\lambda) \\ G_1(\lambda) & G_2(\lambda) \end{vmatrix} = 0 \tag{20}$$

in order to satisfy the outer boundary conditions with exponential speed for any (β, λ). Then by carrying out the numerical integration of (15) subject to the conditions outlined above we can calculate the constants C_1, C_2, G_1 and G_2 subject to (20) and so compute λ. This was effected by using Newtons' method following the shooting schemes of [14]. We remark that, throughout the dual solutions domain of β, the *smallest* value λ_l is always negative on the lower-branch while it assumes positive values on the upper-branch. This behaviour confirms that the lower-branch solution is unstable while that of the upper-branch is stable as manifested by the numerical solutions of (12) subject to (13). Of these solutions, that corresponding to $\beta = 1$ represents a new exact solution of the Navier–Stokes equations.

It is worth noting here that Eqs (4a) used to determine the bifurcation point β_b are the same as Eqs (15) with $\lambda = 0$. In fact the bifurcation and the change in the temporal stability is the manifestation of one and the same phenomenon. According to the factorization theorem [19] $\lambda(\beta)$ must change sign across the regular turning point β_b which means that the solution is stable on one side of the said point and is unstable on the other; i.e. the exchange of stability takes place at the said point. The present results (and those of [12] regarding a similar problem) confirm this behaviour.

The author would like to thank the referees for their helpful comments and suggestions regarding this work.

References

1. L.G. Loitsianskii, Interference of boundary layers, *CAHI, Moscow, Report* No. 249 (1936) (English translation ARC 3186-FM342).
2. L.G. Loitsianskii and V.P. Bolshakov, On motion of fluid in a boundary layer near line of intersection of two planes. *CAHI, Moscow, Report* No. 279 (1936) (English translation NACA-TM-1308).
3. J.R.A. Pearson, Homogeneous turbulence and laminar viscous flow. Ph.D. Thesis, Cambridge University (1957).
4. S.G. Rubin, Incompressible flow along a corner. *J. Fluid Mech.* 26 (1966) 97–110.
5. S.G. Rubin and B. Grossman, Viscous flow along a corner: numerical solution of the corner-layer equations. *Quart. Applied Math.* 29 (1971) 169–186.
6. W.H. Barclay and A.H. Ridha, Flow in streamwise corners of arbitrary angle. *AIAA J.* 18 (1980) 1413–1420.
7. S.R. Wilkinson and M. Zamir, Cross-flow and vorticity patterns in the corner boundary layer at different corner angle. *Aero. J.* 88 (1984) 309–316.
8. S.S. Desai and K.W. Mangler, Incompressible laminar boundary layer flow along a corner formed by two intersecting planes. *RAE TR* 74062 (1974).
9. F.T. Smith, Three dimensional stagnation point flow into a corner. *Proc. R. Soc. Lond. A* 334 (1975) 489–507.
10. A. Ridha, Sur la couche limite incompressible laminaire le long d'un dièdre. *C.R. Acad. Sci. Paris* t. 311, Série II (1990) 1123–1128.

11. M. Zamir, Similarity and stability of the laminar boundary layer in a streamwise corner. *Proc. R. Soc. Lond. A* 377 (1981) 269–288.
12. J.M. Merkin, On dual solutions occurring in mixed-convection in porous medium. *J. Eng. Math.* 20 (1985) 171–179.
13. T. Mahmood and J.M. Merkin, Similarity solutions in axisymmetric mixed-convection boundary-layer. *J. Eng. Math.* 22 (1988) 73–92.
14. T. Cebeci and H.B. Keller, Shooting and parallel shooting methods for solving the Falkner–Skan boundary-layer equation. *J. Comp. Physics* 7 (1971) 301–309.
15. H.B. Keller and T. Cebeci, An inverse problem in boundary-layer flows: numerical determination of pressure gradient for a given wall shear. *J. Comp. Physics* 10 (1972) 151–161.
16. M. Zamir and A.D. Young. Experimental investigation of the boundary layer in a streamwise corner. *Aero Quart.* 21 (1970) 313–339.
17. W.H. Banks and P.G. Drazin, Perturbation methods in boundary-layer theory. *J. Fluid Mech.* 58 (1973) 763–775.
18. C.I. Christof and A. Ridha, A strongly implicit scheme for conjugated nonlinear boundary-value problems. Submitted to *J. Comp. Physics* for publication (1991).
19. G. Iooss and D.D. Joseph, *Elementary Stability and Bifurcation Theory*. Springer-Verlag (1980).
20. F.T. Smith, Non-uniqueness in wakes and boundary layers. *Proc. R. Lond. A* 391 (1984) 1–26.